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Abstract. A new exact solution to the 'slow rotation' structure equations is presented. It is briefly compared with the slowly rotating limit of the Wahlquist solution, with which the new solution shares an equation of state.

1. Introduction

There has been much recent interest in studying the properties of perfect fluid solutions to the slow rotation (Cohen and Brill 1968, Adams *et al* 1973, 1974, Whitman and Pizzo 1979, Bayin 1981) equations of general relativity. Although these solutions are not solutions to the exact field equations, their study is nonetheless important. All known neutron stars satisfy the conditions of slow rotation. Thus slowly rotating perfect fluid solutions can serve as mathematical models for neutron stars. In addition, due to the extreme difficulty in solving the exact (nonlinear) field equations for a rotating perfect fluid, workers are faced with the choice of studying slowly rotating solutions or focusing their attention elsewhere.

In fact, their difficulty is such that the exact field equations for a rotating perfect fluid have only one known non-cylindrically symmetric solution. This solution, due to Wahlquist (1968), has been known for some time. Unfortunately the Wahlquist solution has a peculiar property which is not well understood. It was determined by its discoverer that the surfaces of constant pressure are prolate rather than oblate. This result is in contrast to one's 'Newtonian-oriented' intuition.

Is the prolateness due to the also somewhat unphysical equation of state

$$\mu + 3p = \text{constant}$$

or perhaps, as Wahlquist suggested, due to the tidal forces exerted by an external distribution of matter? It might be that our intuition fails in these matters. Since a junction with a vacuum solution has not been found, can one be found, or is such a junction impossible? It is believed here that a good understanding of the sole, exact axially symmetric rotating fluid solution is essential.

In this paper we present a new solution to the slowly rotating field equations. This solution is similar to the Wahlquist solution in that both share a common equation of state. The solution differs from the Wahlquist solution in that in the limit of slow rotation the off-diagonal rotation metric functions are different. A comparison between the two solutions is helpful in understanding rotating perfect fluids.

This paper is organised as follows. In § 2 we present the field equations valid for the case of slow rotation. In § 3, we solve the equations for a slowly rotating

Whittaker-like (Whittaker 1968) solution. This represents a new analytic solution to the slow rotation field equations. In § 4 we review the Wahlquist solution and examine its limit of slow rotation. A discussion is given in § 5.

2. The field equations

The conditions of slow rotation lead to the metric

$$ds^2 = \gamma^2(r)\,dt^2 - \tau^{-1}(r)\,dr^2 - r^2[d\theta^2 + \sin^2\theta\,(d\phi - \Omega(r)\,dt)^2]. \tag{1}$$

In this expression, $\gamma(r)$ and $\tau(r)$ are solutions to the equation

$$\tau'(r) - \frac{2(\gamma + r\gamma' - r^2\gamma'')}{r(\gamma + r\gamma')}\tau(r) = \frac{-2\gamma}{r(\gamma + r\gamma')}, \qquad ' \equiv \frac{d}{dr}. \tag{2a}$$

The remaining field equations are ($8\pi G = c = 1$)

$$r^2\mu = 1 - \tau - r\tau', \qquad r^2 p = (\gamma + 2r\gamma')(\tau/\gamma) - 1, \tag{2b, c}$$

$$\tau(\Omega'' + 4\Omega'/r) = \tfrac{1}{2}(p + \mu)r(\Omega' + 4\Omega/r - 4\omega/r), \tag{2d}$$

where $\omega(r)$ is the rotation rate of the star. Given a knowledge of $\omega(r)$, the problem at hand is completely determined by solving the system (2), given an equation of state. It will be noticed that (2*d*) is the only equation in the system involving rotation. One can then use a known static solution to (2*a*)–(2*c*) in (2*d*) and solve for the remaining metric function, $\Omega(r)$, after specifying $\omega(r)$.

We note that (2*d*), since it is a linear ordinary differential equation, represents an improvement, in some sense, over the nonlinear partial differential equations in the exact case. Although the general solution is not known, particular solutions (i.e. solutions for a particular τ, μ, p and ω) are known. In § 3 we present a new solution.

3. Slowly rotating Whittaker-like solution

The Whittaker solution is

$$ds^2 = \phi^{-2}\,dt^2 - [\phi^2/(1 - K^2Gr^2)]\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{3a}$$

where

$$\phi^{-2} = 1 + K^{-2}\{1 - [(1 - K^2Gr^2)^{1/2}/KG^{1/2}r]\sin^{-1}(KG^{1/2}r)\}, \tag{3b}$$

and K and G are constants.

The mass density and pressure are given by

$$3p + \mu = G, \qquad \mu = \tfrac{1}{2}G(3K^2/\phi^2 - 1), \qquad p = \tfrac{1}{2}G(1 - K^2/\phi^2). \tag{4a, b, c}$$

The pressure is a decreasing function, thus the density must be an increasing one. This property is somewhat unphysical. In this case equation (2*d*) becomes

$$(1 - K^2Gr^2)\Omega'' + (4 - \tfrac{9}{2}GK^2r^2)\Omega'/r - 2GK^2(\Omega - \omega) = 0. \tag{5}$$

We look for a solution in the case of rigid rotation, ω = constant. We introduce the following new independent and dependent variables:

$$x = K^2Gr^2, \qquad y = \Omega - \omega, \tag{6a, b}$$

respectively. With these transformations, (5) becomes

$$x(x-1)y''+(\tfrac{11}{4}x-\tfrac{10}{4})y'+\tfrac{1}{2}y=0$$

where the prime represents differentiation with respect to x. This is the hypergeometric differential equation for the function y:

$$x(x-1)y''+[(1+a+b)x-c]y'+aby=0, \tag{7a}$$

where

$$1+a+b=\tfrac{11}{4}, \qquad c=\tfrac{10}{4}, \qquad ab=\tfrac{1}{2}. \tag{7b, c, d}$$

Solving these expressions for a and b yields

$$a=(7+\sqrt{17})/8, \qquad b=(7-\sqrt{17})/8. \tag{8a, b}$$

The general solution of (7a) is then

$$y=A\,{}_2F_1(a,b,c;x)+Bx^{1-c}{}_2F_1(a+1-c,b+1-c,2-c;x), \tag{9}$$

where A and B are arbitrary constants and F is the hypergeometric function. Since

$$1-c=-\tfrac{3}{2}, \tag{10}$$

we must choose the constant $B=0$ in order that y is non-singular at $x=0$. In terms of the original variables

$$\Omega(r)=\omega+A\,{}_2F_1(a,b,c;K^2Gr^2). \tag{11}$$

The hypergeometric function converges for all positive values of $x<1$. The series representation

$${}_2F_1(a,b,c;x)=1+\frac{ab}{c}x+\frac{a(a+1)b(b+1)}{2c(c+1)}x^2+\ldots \tag{12}$$

terminates if a or b is a negative integer or zero. An examination of equations shows that a and b are positive; the series does not terminate.

Requiring continuity of Ω and the first derivative ($\Omega(r)=2Jr^{-3}$ in the exterior):

$$J=\tfrac{1}{2}\omega R^3[1-3F_0/(F_0'R+3F_0)], \qquad \Omega=\omega[1-3F/(F_0'R+3F_0)], \tag{13a, b}$$

where the subscript '0' refers to the value of the hypergeometric function or its derivative evaluated at $r=R$, the boundary of the body. The solution presented in this section represents one of only a handful of solutions to the slow rotation structure equations for $\omega=$ constant, two being given by Whitman and Pizzo (1979), one by Adams and co-workers (Adams *et al* 1973), and the remaining one by Bayin (1981).

4. The Wahlquist solution

The Wahlquist solution represents an axially symmetric, stationary, type D solution of the Einstein field equations for a rigidly rotating perfect fluid with arbitrary angular velocity. Again, the equation of state can be written

$$3p+\mu=\text{constant}=G. \tag{4a}$$

In the limit of slow rotation (i.e. to first order in the rotation parameter), the Wahlquist solution can be written as

$$ds^2 = \phi^{-2}\,dt^2 - [\phi^2/(1-K^2Gr^2)]\,dr^2 - r^2[d\theta^2 + \sin^2\theta(d\phi - \Omega\,dt)^2], \tag{14a}$$

where

$$\phi^{-2} = 1 + K^{-2}\{1 - [(1-K^2Gr^2)^{1/2}/KG^{1/2}r]\sin^{-1}(KG^{1/2}r)\}, \tag{14b}$$

$$\Omega = (a/K^2Gr^2)\{1 - [(1-K^2Gr^2)^{1/2}/KG^{1/2}r]\sin^{-1}(KG^{1/2}r)\}. \tag{14c}$$

a = constant.

A peculiar feature of the Wahlquist solution, previously mentioned, is that the $p = 0$ surface is prolate even in the weak field, slowly rotating limit. The cause of this unusual property has never been determined, to the author's knowledge. It will be shown in § 5 that the cause is almost certainly tidal forces due to an external distribution of matter.

5. Discussion

It would seem reasonable to expect that the limit of slow rotation of an exact solution and the exact solution of the slowly rotating equations with the same equation of state be identical. In §§ 3 and 4 we have introduced two rigidly rotating perfect fluid solutions in which the pressure and density were related by the same equation of state. The two solutions are clearly not equivalent as can be readily seen by examination of equations (11) and (14c). In fact Ω given by (14c) is not a solution to (2d) with ω = constant. Equation (2d) relates the rotation metric function, Ω, of a body composed of density μ and with pressure p, with the rotation rate, ω. If an external source is present, (2d) must be altered to reflect this. Ω given by (14c) then is a solution not to (2d) but to another equation incorporating the effect of the external mass(es). It would seem then that external sources are present in the Wahlquist solution and that junction of the solution to the vacuum is therefore impossible.

Even though (2d) is not appropriate in this case, it is possible to estimate the effect of the external mass (in addition to the prolateness of the object) by substituting (14e) into (2d) and calculating the effective rotation rate. This is found to be

$$\omega_{\text{eff}} = -a\left[\frac{1}{4x^2} + \frac{\sin^{-1}x}{(1-x^2)^{1/2}}\left(\frac{1}{2x} - \frac{1}{4x^3}\right)\right], \qquad x = KG^{1/2}r.$$

In table 1 some values of the ratio of $\omega_{\text{eff}}(x)$ to the central value $\omega_{\text{eff}}(x = 0)$ are given. $\omega_{\text{eff}}(x)$ is readily seen to be non-singular for all values of $x < 1$. We note that $\omega_{\text{eff}}(x) \geqslant \omega_{\text{eff}}(0)$ and that $\omega'_{\text{eff}}(x) \geqslant 0$. Thus the fluid near the surface ($p = 0$) of the body is affected to a greater extent by the external masses than the fluid near the centre. This is no surprise. Since the ratio is almost equal to 1 for $x < 0.35$, the Wahlquist solution does not differ greatly from the slowly rotating solution in this region. However for $x > 0.7$, the two solutions are very different.

The slowly rotating Whittaker solution found in this paper represents only the fifth such analytic solution to the field equations (2a)–(2d) with ω = constant. Since the density increases with the radial distance, the solution is appropriate for modelling regions in neutron stars in which density inversions take place. This property is also present in a class of the slowly rotating solutions given by Bayin.

Table 1.

$\omega_{eff}(x)/\omega_{eff}(x=0)$	x	$\omega_{eff}(x)/\omega_{eff}(x=0)$	x
1	0	1.186	0.50
1.002	0.05	1.238	0.55
1.006	0.10	1.301	0.60
1.014	0.15	1.381	0.65
1.025	0.20	1.483	0.70
1.039	0.25	1.618	0.75
1.058	0.30	1.803	0.80
1.081	0.35	2.077	0.85
1.110	0.40	2.533	0.90
1.144	0.45	3.538	0.95

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